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MISSION ANALYSIS REPORT - MAR 4

THE GRAVITATIONAL POTENTIAL OF A TRIAXIAL EARTH

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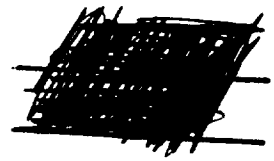

BY

C. A. WAGNER

ADVANCED MISSIONS AND RESEARCH SECTION

SPACECRAFT SYSTEMS AND PROJECTS DIVISION

OCTOBER 31, 1962

GODDARD SPACE FLIGHT CENTER

GREENBELT, MD.

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4-16718

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INTRODUCTION AND SUMMARY 16718

In most references, for its great convenience in calculations, the spheroidal earth's exterior gravity potential is presented as a series of spherical harmonic terms. In the first part of this paper, that series is derived in full detail as a classical solution of Laplace's equation in spherical coordinates. In the second part of the report, the series is truncated at the $1/r^3$ term as is often done in the literature. It is shown that this truncation gives the potential for a rotating triaxial earth(having small oblateness and equatorial eccentricity)with an equipotential surface. The coefficients of the truncated series are interpreted as functions of the geometric parameters of the matching triaxial earth model and it's rotation. The physical significance of the harmonic coefficients as measured to date from gravity surveys and satellite data, developed according to the theory presented in this paper gives:

1. An earth oblateness of 1/298.2 ,
2. An equatorial eccentricity of 7.2×10^{-6} which corresponds to a a difference of major and minor equatorial diameters of about 300 feet.

In addition to the harmonics contributing principally to the triaxial earth, the higher order harmonics are interpreted, in the third part of the paper, as being due to additional small mass deviations in longitude and lattitude from the surface of this triaxial earth. The effect on the model of terms through J44 is calculated from a geopotential due to Kozai(1962). The presence of the J_{31} harmonic coefficient as reported by Kozai is interpreted as arising from a maximum surface deviation of 70 feet at latitude $\pm 70^{\circ}$ and longitude 5° east and 175° west of greenwich. This is the greatest deviation due to any single higher harmonic.

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NOMENCLATURE

∇^2 : Differential operator signifying $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in x, y, z rectangular coordinates, for example

Q_i : Metric coefficient of the coordinate q_i. (dimensionless)

q_i : General orthogonal coordinate

ds_i : A small line element in the direction of increasing q_i only. (dimensions of q_i)

r, L, λ : Spherical coordinates of radius, latitude and longitude with respect to a principal axis of inertia of the mass distribution.

V : The gravitational potential. (units of energy/mass)

R, H, Φ : Separated variations of the potential, functions of radius, latitude and longitude respectively.

d : The total differentiation operator.

C_n^m : Intermediate potential constants. (units of energy/mass)

P_n^m : The Associated Legendre Function of order n, power m.

P_n^0 or P_n : The Legendre Polynomial of order n.

F_n^m : Intermediate potential constants. (units of energy/mass)

λ_{nm} : Longitude of the principle plane of symmetry corresponding to the nm harmonic of the potential of the mass distribution, with respect to a principal axis of inertia of the mass distribution.

G : The universal gravitational constant. (6.673×10^{-8} dyne-cm^2/gm^2)

M : The total mass of the distribution whose potential is desired.

I_0 : The Moment of Inertia of the distribution about the origin.

I_r : The Moment of Inertia of the distribution about the line vector r to the test point in the field.

x_i : Rectangular coordinates in inertial space aligned to the axes of principal moment of inertia of the mass distribution.

A, B, C : Principal moments of inertia of the mass distribution.

J_{nm} : Potential constant for the nm harmonic of the potential of the mass distribution. (units of energy/mass)

$\bar{a}$: The average of major and minor radii of the model earth ellipsoid's equator.

γ : The polar flattening or oblateness coefficient of the model earth ellipsoid. (dimensionless)

e : The eccentricity of the model earth ellipsoid's equator.

NOMENCLATURE-(Cont.)

a, b, c : Half axes of the model earth ellipsoid. (c is the polar radius)

R : In addition to the potential variation in radius; the radius to the surface of the model earth ellipsoid.

w : The earth's rotation rate.(units of angular velocity)

C_0 : The constant amount of the potential at the surface of the model earth ellipsoid.

α : Centrifugal potential constant.of the earth.

ϕ : Geographical longitude measured east from the greenwich meridian.

ϕ_{nm} : Geographic longitude of the principal plane of symmetry corresponding to the nm harmonic of the potential of the mass distribution.

Δr : Deviation in radius from the surface of the average earth sphere of radius r to an equipotential surface .

∂ : Partial differentiation operator.

K_0 , λ_0 : Intermediate potential constants.

$+$: Plus.

"B" : The domain of the mass distribution.

Λ : Inertial longitude with respect to the Vernal Equinox.

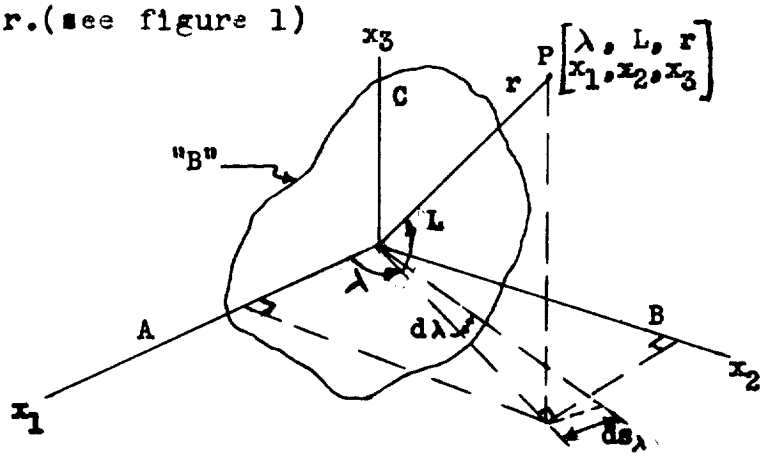
Ψ : Geographic longitude referred to the minor axis of the earth's elliptical equator.

Part One: Derivation of the exterior gravitational potential of a general spheroidal mass distribution.

It is known from the theory of the potential (Ref. 1, pg. 33) that the potential function V of a finite body at points outside the body satisfies Laplace's Equation:

$$\nabla^2 V = 0. \tag{1}$$

Consider body "B" with the origin of instantaneous inertial x_1, x_2, x_3 coordinates inside "B", and exterior point "P" specified by spherical coordinates λ, L, r. (see figure 1)



(Figure 1)

In general orthogonal coordinates q_1, q_2, q_3;

$$\nabla^2 V = \frac{1}{Q_1 Q_2 Q_3}\left\{\frac{\partial}{\partial q_1}\left[\frac{Q_2 Q_3}{Q_1}\frac{\partial V}{\partial q_1}\right] + \frac{\partial}{\partial q_2}\left[\frac{Q_1 Q_3}{Q_2}\frac{\partial V}{\partial q_2}\right] + \frac{\partial}{\partial q_3}\left[\frac{Q_1 Q_2}{Q_3}\frac{\partial V}{\partial q_3}\right]\right\}, \tag{2}$$

where $ds_1 = Q_1 dq_1$ = a small line element in the direction of increasing q_1 coordinate at P. (all other coordinates constant) (See Ref. 2, pgs. 173&175)

In the spherical coordinates of Fig. 1, we identify:

$ds = r \cos L \, d\lambda$, implying $Q_\lambda = r \cos L$

$ds_L = r \, dL$, implying $Q_L = r$

$ds_r = 1. \, dr$, implying $Q_r = 1$

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With these identifications in (2), (1) becomes;

$$\frac{1}{r^2\cos L}\left\{\frac{\partial}{\partial\lambda}\left[\frac{1}{\cos L}\frac{\partial V}{\partial\lambda}\right] + \frac{\partial}{\partial L}\left[\cos L\,\frac{\partial V}{\partial L}\right] + \frac{\partial}{\partial r}\left[r^2\cos L\,\frac{\partial V}{\partial r}\right]\right\} = 0.$$

Or, expanding;

$$\frac{1}{r^2\cos L}\left\{\frac{1}{\cos L}\,\frac{\partial^2 V}{\partial\lambda^2} + \frac{\partial}{L}(\cos L\,\frac{\partial V}{\partial L}) + \cos L\,\frac{\partial}{\partial r}\,(r^2\,\frac{\partial V}{\partial r})\right\} = 0.$$

Or:

$$\frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2\,\frac{\partial V}{\partial r}) + \frac{1}{r^2\cos L}\,\frac{\partial}{\partial L}\,(\cos L\,\frac{V}{L}) + \frac{1}{r^2\cos^2 L}\,\frac{\partial^2 V}{\partial L^2} = 0. \qquad (3)$$

To separate variables, assume;

$$V = R(r)\,H(L)\,\Phi(\lambda). \qquad (4)$$

Substituting (4) into (3), performing the indicated partial derivatives and rearranging, (3) becomes;

$$\frac{H\Phi}{r^2}\,\frac{\partial}{\partial r}\,(r^2R') + \frac{R\Phi}{r^2\cos L}\,\frac{\partial}{\partial L}\,(\cos L\,H') + \frac{R\,H}{r^2\cos^2 L}\,\Phi'' = 0. \qquad (5)$$

The primes denote derivatives with respect to the corresponding independent variables.

Multiplying (5) by $r^2\cos^2 L$, gives;

$$H\Phi\cos^2 L\,\frac{\partial}{\partial r}\,(r^2R') + R\,\Phi\cos L\,\frac{\partial}{\partial L}(\cos L\,H') + R\,H\,\Phi'' = 0. \qquad (6)$$

Dividing (6) through by $R\Phi H$ isolates $\Phi(\lambda)$. (6) becomes;

$$\frac{\cos^2 L}{R}\,\frac{\partial}{\partial r}\,(r^2R') + \frac{\cos L}{H}\,\frac{\partial}{\partial L}\,(\cos L\,H') = -\frac{\Phi''}{\Phi} = +m^2. \qquad (7)$$

Small m is a constant, (to be determined later) since the left hand side of (7) is a function of r and L only, while the right $\left(-\frac{\Phi''}{\Phi}\right)$ is a function of λ only.

From (7); $\Phi'' + m^2\Phi = 0$

implying; $\Phi(\lambda) = K_0 \cos m(\lambda - \lambda_0)$, (8)

where K_o and λ_o are constants of integration.

Dividing (7) by $\cos^2 L$ and rearranging allows one to isolate H and R. (7) thus becomes;

$$\frac{1}{H\cos L}\frac{\partial}{\partial L}(\cos L\, H') - \frac{m^2}{\cos^2 L} = -\frac{1}{R}\frac{\partial}{\partial r}(r^2 R') = -f. \tag{9}$$

Small f is a constant(to be determined later) since the left hand side of (9) is a function of L only and the right is a function of r only.

From (9);

$$\frac{1}{R}\frac{\partial}{\partial r}(r^2 R') = f, \tag{10}$$

And also;

$$\frac{1}{H\cos L}\frac{\partial}{\partial L}(\cos L\, H') - \frac{m^2}{\cos^2 L} = -f. \tag{11}$$

We transform the independent variable in (11) from L to x by setting $x = \sin L$.

Thus; $\frac{d}{dL} = \frac{d}{dx}\frac{dx}{dL} = \cos L\frac{d}{dx}$: (12)

And, $\cos L = (1-x^2)^{\frac{1}{2}}$, $\cos^2 L = 1-x^2$.

In (10) and (11) we let $f = n(n+1)$, with n a constant to be determined later.

With these changes, (11) goes to;

$$\frac{1}{H}\frac{d}{dx}\left[(1-x^2)\frac{dH}{dx}\right] - \frac{m^2}{1-x^2} + n(n+1) = 0. \tag{13}$$

Note that the partial differentiation in (11) is equivalent to total since only one independent variable (L) is involved.

Expanding (13) and multiplying through by H gives;

$$(1-x^2)\frac{d^2H}{dx^2} - 2x\frac{dH}{dx} + \left[n(n+1) - m^2/1-x^2\right]H = 0. \tag{14}$$

(14) is recognized as the Associate Legendre Equation. It has applicable solutions;

$$H_{nm} = C_n^m (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m}. \tag{15}$$

The $P_n(x)$ are the well known Legendre Polynomials,(See Ref. 3, pgs. 305, 306, 312 and 313) and m and n are non-negative integers; $m \leq n$. Other solutions of (14) are possible when n is or is not an integer, if $|x| \neq 1$. But $|x| = |\sin L| \leq 1$ in this formulation of the Associate Legendre Equation and the other solutions are not admissible. They give infinities for $|x| = 1$. (See ref. 4, pgs. 192-194 and 199)

The polynomials $P_n(x)$ take the form:

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\left\{x^n - \frac{n(n-1)x^{n-2}}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)x^{n-4}}{2.4(2n-1)(2n-3)} - \ldots\right\}. \quad (16)$$

In particular, the first six polynomials are:

$$P_0(x) = P_0(\sin L) = 1$$
$$P_1(x) = P_1(\sin L) = \sin L$$
$$P_2(x) = P_2(\sin L) = \tfrac{1}{2}(3\sin^2 L - 1)$$
$$P_3(x) = P_3(\sin L) = \tfrac{1}{2}(5\sin^3 L - 3\sin L) \quad (17)$$
$$P_4(x) = P_4(\sin L) = (1/8)(35\sin^4 L - 30\sin^2 L + 3)$$
$$P_5(x) = P_5(\sin L) = (1/8)(63\sin^5 L - 70\sin^3 L + 15\sin L)$$
$$P_6(x) = P_6(\sin L) = (1/16)(231\sin^6 L - 315\sin^4 L + 105\sin^2 L + 3)$$

To determine the variation in r, in eqauation (10) we set $f = n(n+1)$ and expand:

$$r^2R'' + 2rR' - n(n+1) = 0. \quad (18)$$

(18) is a "Cauchy" Equation, (See ref. 4, pg. 199) whose general solution is;

$$R(r) = A_1r^n + A_2r^{-(n+1)}. \quad (19)$$

If (as is conventional) we arbitrarily stipulate that $V \to 0$ as $r \to \infty$, then A_1 must be set equal to zero to avoid the infinities from the first term in (19) for $n > 0$; and to avoid a finite V as $r \to \infty$ for $n = 0$.

Combining the L, λ and r variations in (8), (15) and (19); formulating new coefficients F_n^m and λ_{nm}, gives the series solution of

Laplace's Equation in spherical coordinates as:

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} F_n^m r^{-(n+1)} \cos^m L \, \frac{d^m P_n(\sin L)}{d(\sin L)^m} \cos m(\lambda - \lambda_{nm}) . \qquad (20)$$

The potential field of (20) is usually separated into zonal ($m = 0$) and sectorial-tesseral terms ($m \neq 0$):

$$V = \sum_{n=0}^{\infty} F_n^0 \, r^{-(n+1)} P_n(\sin L) + \ldots$$
$$\ldots + \sum_{n=1}^{\infty} \sum_{m=1}^{n} F_n^m \, r^{-(n+1)} \cos^m L \, \frac{d^m P_n(\sin L)}{d(\sin L)^m} \cos m(\lambda - \lambda_{nm}) . \qquad (21)$$

The sectorial terms of (21) are those of $m \neq 0$ for which $n = m$. The tesseral are those of $m \neq 0$ for which $n \neq m$. The λ_{nm} represent the location in the space (λ, L, r) of the principle plane of longitudinal symmetry for the nm harmonic ($m \neq 0$).

Part Two: Matching The Gravitational Potential To An Equivalent Earth Model.

Consider the series (21) terminated at $n = 2$ so that;

$$V = (1/r)(F_0^0) + (1/r^2)\left[F_1^0 \sin L + F_1^1 \cos L \cos(\lambda - \lambda_{11})\right] + (1/r^3)\left[\ldots \right.$$
$$\left. \ldots F_2^0 (3\sin^2 L - 1) + \frac{3F_2^1}{2} \sin 2L \cos 2(\lambda - \lambda_{22})\right] + \text{order}(1/r^4) \qquad (22)$$

Consider the potential of a finite continuous distribution of matter at a point sufficiently outside the distribution that the radius from the origin of coordinates to the field point is greater than the radius to any part of the distribution. If the origin of coordinates is taken at the center of mass(c.m.) of the distribution, it is shown (ref. 1, pg. 84) that;

$$V/G = M/r + \left(\frac{2I_0 - 3I_r}{2r^3}\right) + \text{order}\,(1/r^4) , \qquad (23)$$

M is the total mass of the distribution, I_0 is the moment of inertia

about the origin, I_r is the moment of inertia about the line vector $\vec{r}$ to the field point, and G is the universal gravitational constant.

The discussion of the physical significance of the coefficients of (22) is simplified by considering axes x_1, x_2, x_3 oriented along the directions of the principle moments of inertia A, B and C of the mass distribution. (See fig. 1)

Considering the field as represented by the mass-integrals of (23), we see that the expansion in harmonics of (22) is equivalent to the mass integral harmonic series of (23) if and only if:

$$F_0^0 \equiv M \,, \tag{24}$$

$$(1/G)\left[F_1^0 \sin L + F_1^1 \cos(\lambda - \lambda_{11})\right] \equiv 0 \,, \tag{25}$$

$$(F_2^0/2)(3\sin^2 L - 1) + (3F_2^1/2)\left[\sin 2L \cos(\lambda - \lambda_{21})\right] + \ldots$$
$$\ldots 3F_2^2 \cos^2 L \cos 2(\lambda - \lambda_{22}) \equiv (G)\frac{(2I_0 - 3I_r)}{2} \tag{26}$$

Etc...

Condition (24) gives the familiar GM/r potential term which, if it were the only term of the series present, would state that the whole mass M acted effectively at the c.m. Condition (25) implies that $F_1^0 \equiv F_1^1 \equiv 0$ since the two terms are linearly independent.

We define the following new constants:

$$F_{20}^1 \equiv \frac{3F_2^1}{2G}\cos\lambda_{21} \,, \quad F_{21}^1 \equiv -\frac{3F_2^1}{2G}\sin\lambda_{21} \,, \quad F_{20}^2 \equiv \frac{F_2^2}{G}\cos 2\lambda_{22} \,,$$

$$F_{20}^0 \equiv \frac{F_2^0}{G} \,, \quad F_{21}^2 \equiv -\frac{F_2^2}{G}\sin 2\lambda_{22} \,.$$

To facilitate the term by term matching of (26), we transform back to rectangular coordinates x_1, x_2, x_3, where;

$$x_1 \equiv r \cos L \cos\lambda$$
$$x_2 \equiv r \cos L \sin\lambda \tag{27}$$
$$x_3 \equiv r \sin L$$

Expanding (26), introducing the new constants and the transformation of (27), (26) becomes;

$$\frac{F^0_{20}}{2}\left[\frac{3(x_3)^2 - (x_1)^2 + (x_2)^2 + (x_3)^2}{r^2}\right] + \frac{2F^1_{20}(x_1)(x_3)}{r^2} + \cdots$$

$$\cdots \frac{2F^1_{21}(x_2)(x_3)}{r^2} + 3F^2_{20}\frac{\left[(x_1)^2 - (x_2)^2\right]}{r^2} + \frac{6F^2_{21}(x_1)(x_2)}{r^2} = \frac{2I_0 - 3I_r}{2} \quad .$$

But, from their definitions;

$$2I_0 = (A + B + C)(x_1)^2 + \left[(x_2)^2 + (x_3)^2\right]/r^2 \quad , \text{ and}$$

$$I_r = \left[A(x_1)^2 + B(x_2)^2 + C(x_3)^2\right]/r^2 \quad .$$

Condition (26) thus becomes;

$$(x_1)^2\left(-\frac{F^0_{20}}{2} + 3F^2_{20}\right) + (x_2)^2\left(-\frac{F^0_{20}}{2} - 3F^2_{20}\right) + (x_3)^2(F^0_{20}) + (x_1)(x_2)(6F^2_{21}) +$$

$$\cdots(x_1)(x_3)(2F^1_{20}) + (x_2)(x_3)(2F^1_{21}) = (x_1)^2\left[\frac{(A + B + C)}{2} - \frac{3A}{2}\right] + \cdots$$

$$\cdots(x_2)^2\left[\frac{(A + B + C)}{2} - \frac{3B}{2}\right] + (x_3)^2\left[\frac{(A + B + C)}{2} - \frac{3C}{2}\right] \quad . \qquad (28)$$

Since all the terms on either side of the equality in (28) are linearly independent, the only way we can satisfy (28) identically for all sets of x_1, x_2, x_3 for non zero coefficients, is for;

$$-\frac{F^0_{20}}{2} + 3F^2_{20} = (B + C - 2A)/2 \qquad (29)$$

$$-\frac{F^0_{20}}{2} - 3F^2_{20} = (A + C - 2B)/2 \qquad (30)$$

$$F^0_{20} = (A + B - 2C)/2 \qquad (31)$$

$$6F^2_{21} = 0 \qquad (32)$$

$$2F^1_{20} = 0 \qquad (33)$$

$$2F^1_{21} = 0 \quad . \qquad (34)$$

Thus, (32), (33) and (34) determine: $F^2_{21} = F^1_{20} = F^1_{21} = 0$.

Substituting F^0_{20} from (31) into (29) determines F^2_{20} as;

$$F^2_{20} = \frac{B - A}{4} \quad . \tag{35}$$

It may be verified that (30) is satisfied identically with F^2_{20} and F^0_{20} given from (35) and (31).

Therefor, with respect to (r, L, λ) referred to the principle axes of the mass distribution, the potential (22) becomes;

$$V = \frac{GM}{r} + \frac{G}{r^3}\left[\frac{F^0_{20}}{2}(3\sin^2 L - 1) - 3F^2_{20}\cos^2 L \cos 2\lambda\right] + \text{order}(1/r^4) \quad .$$

In terms of the coefficients of ref. 6, this may be written as;

$$V = \frac{GM}{r}\left[1 - \frac{J_{20}}{2}(\mathfrak{a}/r)^2(3\sin^2 L - 1) - 3J_{22}(\mathfrak{a}/r)^2\cos^2 L \cos 2\lambda\right] + \text{order}(1/r^4) \tag{36}$$

In (36);

$$J_{20} = -F^0_{20}/M(\mathfrak{a})^2 = \frac{2C - (A + B)}{2M(\mathfrak{a})^2} \quad . \tag{37}$$

$$J_{22} = -F^2_{20}/M(\mathfrak{a})^2 = \frac{A - B}{4M(\mathfrak{a})^2} \quad . \tag{38}$$

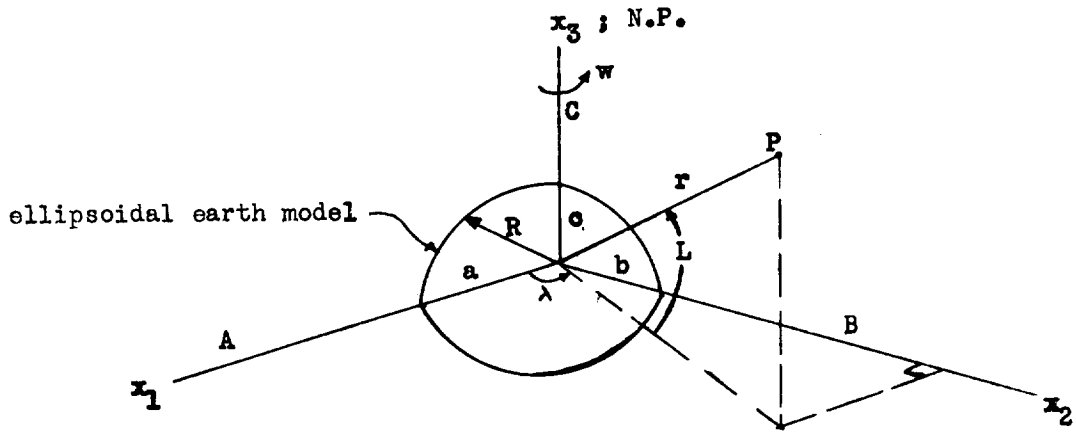
The constant $\mathfrak{a}$ will be defined later as a mean equatorial radius of the model earth ellipsoid.

The gravitational potential of (36) with the higher order terms neglected and a centrifugal term added will be shown to give essentially an equipotential ellipsoidal surface if J_{20} and J_{22} are sufficiently small. J_{20} and J_{22} may then be determined as functions of the rotation, oblateness and ellipticity of a model earth with an equipotential surface; such as would be defined by an equivalent mean sea level for example.

Two questions remain to be resolved before giving geometric significance to J_{20} and J_{22}. The first concerns the validity of ignoring the $1/r^2$ term in the gravitational potential for those field points (i.e. on surface of an ellipsoidal earth away from the equatorial major axis) for which some of the mass distribution is farther from the origin then the

field point. The infinite series of harmonic functions in (20) is the general solution of Laplace's Equation with the boundary condition $V(r \to \infty) = 0$, and represents the potential at all points exterior to the surface of the mass distribution. But at points more strongly exterior to the surface, the direct integration by series for the potential shows that the $1/r^2$ term is absent for the choice of origin at the c.m. The coefficient of the $1/r^2$ term in the harmonic series of (20) is a function of L and λ only. On a radial line, therefor, it is constant. Since it is zero in the outer region, it is zero everywhere to the surface because the inner region to the surface can be completely covered by radial lines from the enfolding outer region.

The second question concerns the orientation of x_1, x_2, x_3 principle axes of inertia in the earth. Let us assume that we spin our ellipsoidal earth model about an arbitrary axis in inertial space. From ref. 5 pp. 377-78 it is shown that, for a rigid body, such an initial rotation is invariable in the absence of external torques if and only if that initial spin axis is a principle axis of inertia. Astronomical observation shows that, to a high degree of accuracy, the earth's north pole spin axis is invariable in direction in inertial space after accounting for all external torques. We can thus arbitrarily (between x_1, x_2 and x_3) identify x_3 as the polar axis of rotation of the model rigid ellipsoidal earth with a thin sea level surface. In the model, then, L corresponds to geocentric latitude with respect to the c.m. λ is the geocentric longitude measured clockwise from the axis x_1 of the equatorial ellipse corresponding to moment of inertia A. The semi-major axis length along x_1 is a. (See figures 1 and 2)



(Figure 2)

In the ellipsoidal earth model of fig. 2, we define e as the ellipticity of the equator and γ as the polar flattening. In terms of a, b, and c, the half axes of the ellipsoid, we define;

$$e = 1 - b/a = (a - b)/a \ , \tag{39}$$

$$\gamma = 1 - c/\bar{a} = (\bar{a} - c)/\bar{a} \ , \tag{40}$$

where; $\bar{a} = (a + b)/2$. (41)

The surface of the ellipsoid in fig. 2 is given by:

$$(x_1/a)^2 + (x_2/b)^2 + (x_3/c)^2 = 1. \tag{42}$$

In terms of the coordinate transformation of (27), the surface is also given by;

$$\begin{aligned} x_1 &= R \cos L \cos\lambda \\ x_2 &= R \cos L \sin\lambda \\ x_3 &= R \sin L \ , \end{aligned} \tag{43}$$

where R is the radius to the surface of the ellipsoid.

Combining (42) and (43):

$$\frac{\cos^2 L \cos^2\lambda}{a^2} + \frac{\cos^2 L \sin^2\lambda}{b^2} + \frac{\sin^2 L}{c^2} = 1/R^2 \ . \tag{44}$$

Multiplying (44) through by b^2, solving for b/a from (39) and rearranging, (44) becomes;

$$(1 - e)^2(\cos^2 L \cos^2\lambda) + \cos^2 L \sin^2\lambda = b^2\left(\frac{1}{R^2} + \frac{\sin^2 L}{c^2}\right) . \tag{45}$$

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Solving for and introducing c^2 from (40) into (45), transforms (45) to;

$$(1 - e)^2\cos^2 L\cos^2\lambda + \cos^2 L \sin^2\lambda = \frac{b^2}{\bar{a}^2}\left[\frac{\bar{a}^2}{R^2} - \frac{\sin^2 L}{(1-\gamma)^2}\right] \quad . \qquad (46)$$

Combining (39) and (41), we can write;

$$\frac{b^2}{\bar{a}^2} = \left[\frac{2(1 - e)}{2 - e}\right]^2 \quad . \qquad (47)$$

Substituting (47) into (46) and rearranging, (46) becomes;

$$(\bar{a}/R)^2 = \frac{(1-e)^2\cos^2 L\cos^2\lambda + \cos^2 L\sin^2\lambda + 2(1-e)\sin^2 L/(2-e)(1-\gamma)^2}{\left[\frac{2(1-e)}{2-e}\right]^2} \quad . \qquad (48)$$

Simplifying, (48) becomes;

$$(\bar{a}/R)^2 = (1/4)(2-e)^2\cos^2 L\cos^2\lambda + \frac{(1/4)(2-e)^2\cos^2 L\sin^2\lambda}{(1-e)^2} + \frac{\sin^2 L}{(1-\gamma)^2} \quad . \qquad (49)$$

Expanding;

$$\left[\frac{2-e}{1-e}\right]^2 = (4-4e+e^2)(1+2e+3e^2+\ldots) \doteq 4+4e+5e^2 \quad , \text{ for } e<<1. \qquad (50)$$

$$(1-\gamma)^{-2} \doteq 1 + 2\gamma + 3\gamma^2 \quad , \text{ for } \gamma<<1. \qquad (51)$$

With the approximations of (50) and (51) in (49), (49) becomes;

$$(\bar{a}/R)^2 = (1/4)(2-e)^2\cos^2 L\cos^2\lambda + (1/4)(4+4e+5e^2)\cos^2 L\sin^2\lambda + \ldots$$
$$\ldots(1+2\gamma+3\gamma^2)\sin^2 L \quad . \qquad (52)$$

Equation (52) describes the surface of the triaxial ellipsoid to e^2 and γ^2 in the ellipticity and oblateness(polar flattening).

The centrifugal potential at the ellipsoid's surface is given as;

$$V_e = \frac{w^2}{2}\left[(x_1)^2 + (x_2)^2\right]_R = \frac{w^2}{2}\left[R^2\cos^2 L\cos^2\lambda + R^2\cos^2 L\sin^2\lambda\right],$$
$$= \frac{w^2}{2}R^2\cos^2 L \quad . \qquad (53)$$

Small w is the earth's rotation rate.

We define:

$$\alpha = \frac{w^2\bar{a}^3}{GM} \quad . \qquad (54)$$

Combining (36), (53) and (54), the full potential at the surface of the ellipsoid is;

$$V(\text{total}) = V + V_c = \frac{GM}{\bar{a}}\left[\frac{\bar{a}}{R} + \frac{\alpha}{2}(R/\bar{a})^2\cos^2 L - \frac{J_{20}}{2}(\bar{a}/R)^3(3\sin^2 L - 1) + \ldots \right.$$
$$\left. \ldots - 3J_{22}(\bar{a}/R)^3\cos^2 L\cos 2\lambda\right]. \qquad (55)$$

In (55), the higher order harmonics have been neglected.

Assuming that the surface of the earth ellipsoid is a level potential surface of constant amount C_0;

$$C_0 = \frac{GM}{\bar{a}}\left[\frac{\bar{a}}{R} + \frac{1}{2}(R/\bar{a})^2\cos^2 L - \frac{J_{20}}{2}(\bar{a}/R)^3(3\sin^2 L - 1) + \ldots \right.$$
$$\left. \ldots - 3J_{22}(\bar{a}/R)^3\cos^2 L\cos 2\lambda\right] . \qquad (56)$$

Simplifying (54);

$$R = \bar{a}\left[1 + (-e + \frac{e^2}{4})\cos^2 L\cos^2\lambda + (e + \frac{5e^2}{4})\cos^2 L\sin^2\lambda + (2\gamma + 3\gamma^2)\sin^2 L\right]^{-\frac{1}{2}}$$
$$\ldots (57)$$

Expanding (57) to first order smallness in e and γ ;

$$R^{-1} = (\bar{a})^{-1}(1 - \frac{e}{2}\cos^2 L\cos^2\lambda + \frac{e}{2}\cos^2 L\sin^2\lambda + \sin^2 L) . \qquad (58)$$

Similarly;

$$R^2 = (\bar{a})^2(1 + e\cos^2 L\cos 2\lambda - 2\sin^2 L) . \qquad (59)$$

Similarly;

$$R^3 = (\bar{a})^3(1 + \frac{3e}{2}\cos^2 L\cos 2\lambda - 3\sin^2 L) . \qquad (60)$$

Sunstituting (58) through (60) into (56) and ignoring the second order product terms in J_{20} , J_{22} , e , γ and α, (56) becomes;

$$C_0 = \frac{GM}{\bar{a}}\left[1 - \frac{e}{2}\cos^2 L\cos 2\lambda + \sin^2 L + \frac{\alpha}{2}(1 - \sin^2 L) - \frac{J_{20}}{2}(3\sin^2 L - 1) + \ldots \right.$$
$$\left. \ldots - 3J_{22}\cos^2 L\cos 2\lambda\right] . \qquad (61)$$

In order for (61) to be an identity for all L and λ, the following conditions must be satisfied:

$$C_0 = \frac{GM}{\bar{a}}(1 + \frac{\alpha}{2} + \frac{J_{20}}{2}) \qquad (62)$$

$$0 = -\frac{e}{2} - 3J_{22} \quad , \tag{63}$$

$$0 = \gamma - \frac{\alpha}{2} - \frac{3J_{20}}{2} \quad . \tag{64}$$

From (63), we can identify;

$$J_{22} = -e/6 \quad . \tag{65}$$

Similarly, from (64), the potential constant J_{20} is identified as;

$$J_{20} = \frac{2\gamma}{3} - \frac{\alpha}{3} \quad . \tag{66}$$

Expressions (65) and (66) give approximate physical interpretation to the harmonic coefficients in the potential of (36), as functions of the small parameters of the model earth ellipsoid and it's rotation.

We are now in a position to describe the geometry of the earth model and identify axes x_1 and x_2 from satellite and geodetically derived values of J_{20} and J_{22}.

According to Kozai (refs. 7 and 8), the best values of J_{20} and J_{22} are;

$$J_{20} = 1.08248\text{x}10^{-3} \quad , \tag{67}$$

$$J_{22} = -1.2\text{x}10^{-6} \quad . \tag{68}$$

We use the constants;(from O'keefe, Eckels and Squires-1959, cited in ref. 6)

$$GM = 3.98618\text{x}10^{5} \text{ km}^3/\text{sec}^2 \tag{69}$$

$$R = 6.378388\text{x}10^{3} \text{ km.}$$

$$w = .7292115\text{x}10^{-4} \text{ rad./sec. (from ref. 9)} \tag{70}$$

Therefor, from (69) and (70) in (54);

$$\alpha = 3.46165\text{x}10^{-3}.$$

Solving for the oblateness(polar flattening) of the ellipsoid model from (64);with the constants of (67) and (70);

$$\gamma = 1.5\text{x}1.08248\text{x}10^{-3} + .5\text{x}3.46165\text{x}10^{-3} = 1/298.2 \cdot \tag{71}$$

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Similarly, solving for the equatorial ellipticity from (63), with J_{22} given in (68);

$$e = 6x1.2x10^{-6} = 7.2x10^{-6} . \qquad (72)$$

We can now solve for the difference in major and minor radii of the elliptical equator of the model earth.

From (39) and (40);

$$a(1-e) - b = 0 , \qquad (73)$$

$$a + b = 2\bar{a} .$$

Solving (73) for a and b;

$$a = 2\bar{a}/2-e , \qquad (74)$$

$$b = 2\bar{a}(1-e)/2-e .$$

Therefor, from (74);

$$a - b = \frac{2\bar{a}\, e}{2-e} = \bar{a}e(1-e/2)^{-1} \doteq \bar{a}e . \qquad (75)$$

With the results of (69) and (72) in (75), the difference in major and minor raddii of the elliptical equator is calculated as 150. feet.

The order of the principle axes of inertia x_1 , x_2 and x_3 is established by the signs of J_{20} and J_{22} in addition to their relative magnitudes. Considering (38); since J_{22} is negative, $\underline{B>A}$. Considering (37); since J_{20} is positive, $2C > (A + B)$. But since $B>A$, therefor $2C > 2A$; or $\underline{C>A}$. x_1 is thus established as the axis of minimum moment of inertia. On the earth ellipsoid (fig. 2), this is the major axis of the equatorial ellipse. [Note also; for $J_{22}<0$, $e>0$ from (63) and therefor $a>b$ from (39).] To establish the order of principle axes x_2 and x_3, consider the earth ellipsoid as homogeneous. Then (from ref. 5 p. 292) ;

$$B/C = \frac{(c^2/a^2) + 1}{(b^2/a^2) + 1} \text{ ; so that if we can show } b/e > 1, \text{ then } C > B.$$

But from (40), $c = \bar{a}(1-\gamma)$. Using this result with that from (74);

$$b/c = (1-e)/(1-e/2)(1-\gamma) \doteq 1 + \gamma - e/2 > 1. \text{ [from (71) and (72).]}$$

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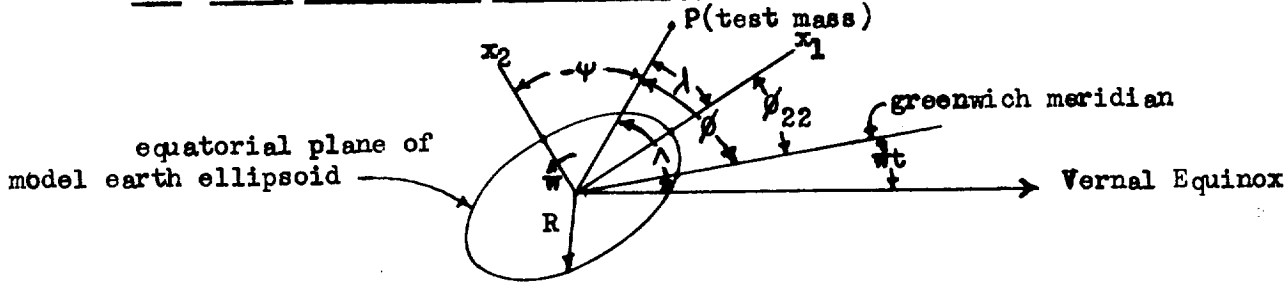
This final result establishes the absolute order of the moments of inertia as;

$$A<B<C. \qquad (76)$$

Forms of the Gravity Potential of the Triaxial Earth:

1. The time dependent potential with respect to inertial space.



(figure 3)

The potential with respect to inertial space will have time dependence in the longitudinal terms. From fig. 3, the longitudinal displacement of the test mass in the equatorial plane is ϕ with respect to the greenwich meridian. If ϕ_{22} is the displacement of the major axis of the equatorial ellipse from greenwich, Λ is the displacement of the test mass from the V. E. . Thus, $\lambda = \phi - \phi_{22} = \Lambda - wt - \phi_{22}$, where w is the earth's rotation rate. Therefor, from (36);

$$V(\text{inertial space}) = \frac{GM}{r}\left[1 - \frac{J_{20}}{2}(\bar{a}/r)^2(3\sin^2 L - 1) + \dots \right.$$
$$\left. \dots - 3J_{22}(\bar{a}/r)^2\cos^2 L\cos 2\{\Lambda - (\phi_{22} + wt)\}\right] \quad , (77)$$

neglecting the higher order harmonics, and with greenwich at the vernal equinox at zero time.

2. The potential with respect to geocentric, model geographical coordinates r, ϕ(east from greenwich) and L(geocentric latitude)

$$V(\text{geocentric-geographical}) = \frac{GM}{r}\left[1 - \frac{J_{20}}{2}(\bar{a}/r)^2(3\sin^2 L - 1) + \dots \right.$$
$$\left. \dots - 3J_{22}(\bar{a}/r)^2\cos^2 L\cos 2(\phi - \phi_{22})\right] \quad , (78)$$

neglecting higher order harmonics.

It is noted that ϕ_{22} locates the major axis of the equatorial ellipse east from greenwich. (See figure 3)

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3. The Potential in Geocentric-Geographical coordinates referred to the Longitude of the minor axis of the earth's equator: (See figure 3)

From figure 3, $\lambda = \frac{\pi}{2} + \psi$, so that, $\cos 2\lambda = \cos 2(\frac{\pi}{2} + \psi) = -\cos 2\psi$. Therefor, with respect to the minor axis of the equatorial ellipse;

$$V(\text{Geocentric-Geographic}) = \frac{GM}{r}\left[1 - \frac{J_{20}}{2}(\bar{a}/r)^2(3\sin^2 L - 1) + \ldots \right.$$
$$\left. \ldots + 3J_{22}(\bar{a}/r)^2\cos^2 L\cos 2\psi\right] , \qquad (79)$$

ignoring higher order harmonics.

Part Three: The physical significance of the presence of the higher order geopotential terms on the earth model.

The form of the potential function in (36) suggests the interpretation that the earth attracts as a sphere,(homogeneous in spherical shells) principally. The higher order terms may be interpreted as arising from small mass deviations in the principal spherical model. At the surface, these deviations are reflected as slight corrugations in latitude, longitude and radius above and below the surface of the average earth sphere. In particular, the previous development has shown how the principal term together with the J_{20} and J_{22} harmonics and the centrifugal potential, describe an earth with an equipotential surface as a triaxial ellipsoid only very slightly deviating from a spherical shape. In the following discussion, the surface deviations from an average earth sphere of radius $\bar{a}$ (chosen for convenience) due to the higher harmonics in turn, will be considered. The deviations will be such as to give equipotential earth surfaces. The complete model is then composed of the higher harmonic deviations superimposed on the triaxial ellipsoid (described by (36) plus the centrifugal potential.) Higher harmonics to the 4th order will be considered. The geopotential through the J_{44} term with respect to Geocentric-Geographic coordinates r, L, ϕ(measured east from greenwich) is:

$$V(\text{Geocentric-Geographical}) = \frac{GM}{r}\Big[1 - (\bar{a}/r)^2 J_{20} P_2^0(\sin L) + \ldots$$

$$\ldots -(\bar{a}/r)^2 J_{22} P_2^2(\sin L)\cos 2(\phi - \phi_{22}) - (\bar{a}/r)^3 J_{30} P_3^0(\sin L) + \ldots$$
$$\ldots - (\bar{a}/r)^3 J_{31} P_3^1(\sin L)\cos(\phi - \phi_{31}) - (\bar{a}/r)^3 J_{32} P_3^2(\sin L)\cos 2(\phi - \phi_{32}) + \ldots$$
$$\ldots - (\bar{a}/r)^3 J_{33} P_3^3(\sin L)\cos 3(\phi - \phi_{33}) - (\bar{a}/r)^4 J_{40} P_4^0(\sin L) + \ldots$$
$$\ldots - (\bar{a}/r)^4 J_{41} P_4^1(\sin L)\cos(\phi - \phi_{41}) - (\bar{a}/r)^4 J_{42} P_4^2(\sin L)\cos 2(\phi - \phi_{42}) + \ldots$$
$$\ldots - (\bar{a}/r)^4 J_{43} P_4^3(\sin L)\cos 3(\phi - \phi_{43}) - (\bar{a}/r)^4 J_{44} P_4^4(\sin L)\cos 4(\phi - \phi_{44})\Big] \quad . \qquad \ldots\ldots(80)$$

The ϕ_{nm} above represent the geographical longitude of the principle plane of longitudinal symmetry for the nm harmonic. (80) is essentially an expansion of (21) in geographic coordinates, with the coefficients rewritten to conform with (36). The P_n^m are the H_{nm} of (15) without the coefficient C_n^m. The $1/r^2$ term in the potential and the J_{21} term have been ignored due to the selection of the reference coordinate system as explained in parts one and two of this report. Depending on the actual choice of geocentric-geographical coordinates of radius and latitude used in a given application, reported coefficents for (80) may vary slightly. For example, if a calculation is to be run with (80) in geographic coordinates referenced to the north pole and an earth center on the equatorial plane, the reported coefficients would appear to be logically in order if the model used to derive those coefficients used the same references. However, since the north pole experiences small anomolous motions which are generally uncorrected for, very small discrepancies may still logically exist between otherwise correct sets of coefficients. More basic still is the consideration that, since the c.m. of the earth is most probably not on the equatorial plane and the uncorrected polar spin axis is definitely not quite an axis of principal moment of inertia for the earth, for calculations using an equatorial center and latitude with respect to the spin pole, a "best" set

of coefficients should include the $1/r^2$ and J_{21} terms.

The potential of (80) is to be referred to an equatorial center and the north polar spin axis.

A) <u>The deviations of the earth's surface giving rise to the J_{30} term of (80)</u>:

To consider the effect on the average sphere of just the J_{30} term, we write the potential of that sphere modified by the J_{30} deviation;

$$V = \frac{GM}{r}\left[1 - (\bar{a}/r)^3 \frac{J_{30}}{2}(5\sin^3 L - 3\sin L)\right] . \qquad (81)$$

According to Y. Kozai in Ref. 8, $J_{30} = -2.6\times10^{-6}$.

We shall reference the deviations from the average sphere at the equator since the deviatory part of (81) goes to zero there. Thus, we let $r = \bar{a}$ when $L = 0$ in (81). The equipotential surface for (81) thus has constant amount $\frac{GM}{\bar{a}}$. The conditional equation for the equipotential surface due to (81) is then;

$$GM/\bar{a} = \frac{GM}{r}\left[1 - (\bar{a}/r)^3 \frac{J_{30}}{2}(5\sin^3 L - 3\sin L)\right] . \qquad (82)$$

Rearranging (82) to solve for r, (82) becomes;

$$(r/\bar{a})^4 - (r/\bar{a})^3 = -\frac{J_{30}}{2}(5\sin^3 L - 3\sin L) . \qquad (83)$$

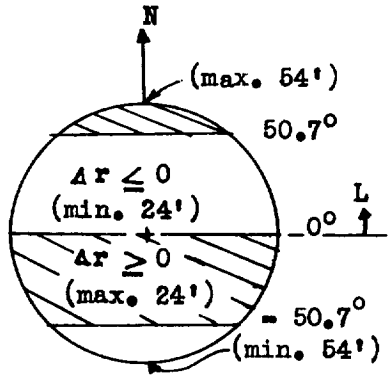
Let $r = \bar{a} + \Delta r$, where $\Delta r \ll \bar{a}$. Then, to the first order in $\Delta r/\bar{a}$, the left side of (83) becomes $\frac{\Delta r}{\bar{a}}$ and (83) is written approximately as;

$$\Delta r \doteq -\frac{\bar{a} J_{30}}{2}(5\sin^3 L - 3\sin L) . \qquad (84)$$

The zeros of (84) are at plus and minus 50.7 degrees latitude.

Furthermore: $\Delta r \leq 0$; $0 \leq L \leq 50.7\text{deg.}$, $-50.7\text{deg.} \leq L \leq -90\text{deg.}$

$\Delta r \geq 0$; Everywhere else. (See figure 4)



(figure 4)
J_{30} Deviations

Extreme radial deviations are attained when;

$$\frac{d(\Delta r)}{dL} = -\frac{\bar{a} J_{30}}{2}(15\sin^2 L\cos L - 3\cos L) = 0. \qquad (85)$$

The zeros of (85) are at L = plus and minus 26.6 degrees, and at L = plus and minus 90 degrees.

Using the value of $\bar{a}$ in (69), and J_{30} as quoted above; Δr extreme, calculated from (84), is equal to;

24 feet (at L = ± 26.6 degrees),

54 feet (at the poles) .

B) Deviations giving rise to the J_{31} term of (80);

From ref. 8 again; $J_{31} = -1.9\times10^{-6}$ and $\phi_{31} = 4.6$ degrees.

The J_{31} geopotential term is;

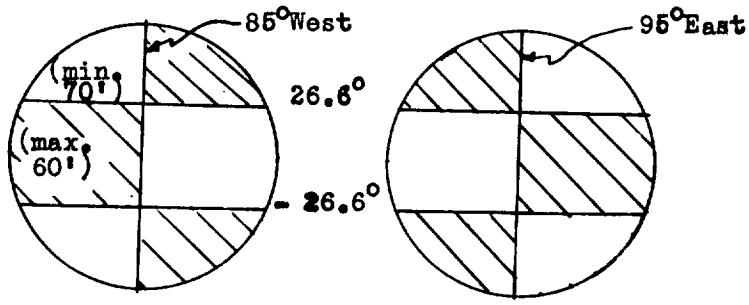
$$-(-1.9\times10^{-6})(\bar{a}/r)^3 \cos L\,(15\sin^2 L - 3)\cos(\phi - 4.6^\circ) .$$

Proceeding as in the case for the deviations giving rise to the J_{30} term, we arrive at radial deviations from the surface of the average sphere of radius $\bar{a}$, given by;

$$\Delta r = -\frac{\bar{a} J_{31}}{2}\cos L(15\sin^2 L - 3)\cos(\phi - 4.6^\circ) \qquad (86)$$

The zeros of (86) are at;

L = ± 90° (all longitudes),
L = ± 26.6° (all longitudes),
ϕ = 94.6° and -85.4° (all latitudes) (see figure 5)



(figure 5)
J_{31} Deviations

Extreme deviations are reached, due to J_{31}, when;

$$\frac{\partial \Delta r}{\partial L} = \cos(\phi - 4.6^\circ)\left[-\sin L(15\sin^2 L - 3) + \cos L(30\sin L)\right] = 0, \text{ and}$$
$$\frac{\partial \Delta r}{\partial \phi} = -\sin(\phi - 4.6^\circ)\ \cos L(15\sin^2 L - 3) = 0. \qquad (87)$$

The simultaneous solution of (87) yields two sets of extreme deviation points across the average sphere. One is at the crossing of the "nodal lines" of figure 5 where Δr is zero. These are evidently flat points of inflection for the equipotential surface due to J_{31}. The other set is at the intersection of:

$\phi = 4.6^\circ$ and -175.4°, with

$L = \pm 70^\circ$ and 0°.

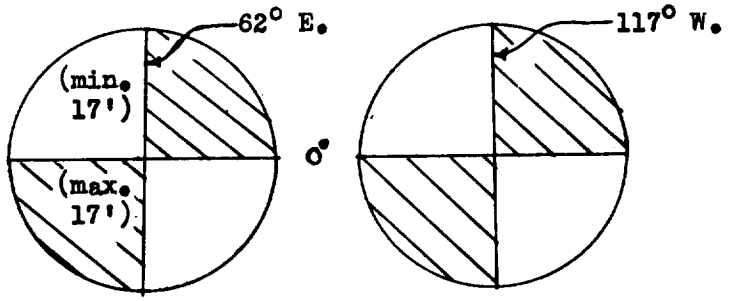
Four of the six extreme points of this set yield the same absolute deviation of 70 feet. The other two yield identical absolute deviations of 60 feet. (see fig. 5)

C) Deviations giving rise to the J_{32} term of (80):

The J_{32} geopotential term is: (from ref. 8)

$$-(-.14\times10^{-6})(\bar{a}/r)^3 15\cos^2 L \sin L \cos 2(\phi + 16.8^\circ) .$$

Deviations are calculated as in B) above, the results appearing on figure 6 below.



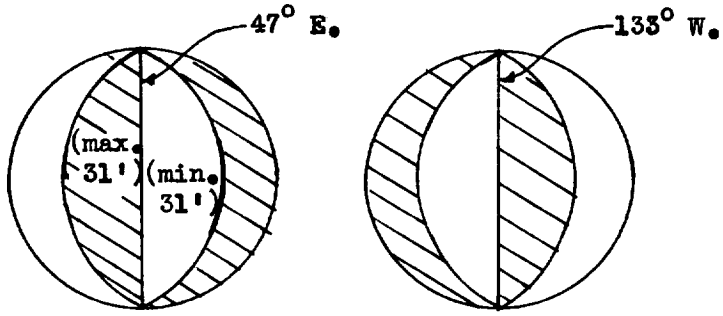
(figure 6)
J_{32} Deviations

D) Deviations giving rise to the J_{33} term of (80):

The J_{33} geopotential term is : (from ref. 8)

$$- (-.10\times10^{-6})(\bar{a}/r)^3 15\cos^3 L \cos 3(\phi - 42.6^\bullet) \quad .$$

Deviations are calculated as in B) above, the results appearing on figure 7 below.



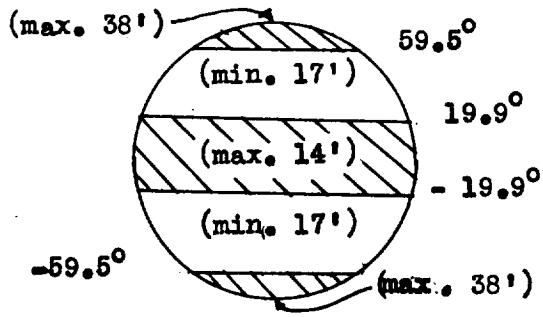
(figure 7)
J_{33} Deviations

E) Deviations giving rise to the J_{40} term of (80):

The J_{40} geopotential term is: (ref. 8)

$$- (-1.84\times10^{-6}(\bar{a}/r)^4 \left[\frac{35}{8}\sin^4 L - 30\sin^2 L + 3\right] \quad .$$

Deviations are calculated as in A) above, the results appearing on figure 8 below.



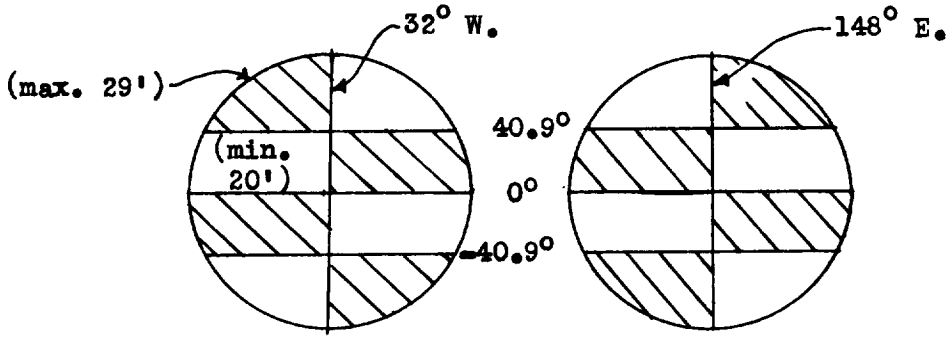
(figure 8)
J_{40} Deviations

F) Deviations giving rise to the J_{41} term of (80):

The J_{41} geopotential term is: (ref. 8)

$$-(-.52\times10^{-6})(\bar{a}/r)^4(1/8)\ 140\sin^3L - 60\sin L\ \cos L\ \cos(\phi - 237.5^\circ)\ .$$

Deviations are calculated as in B) above, the results appearing on figure 9 below.



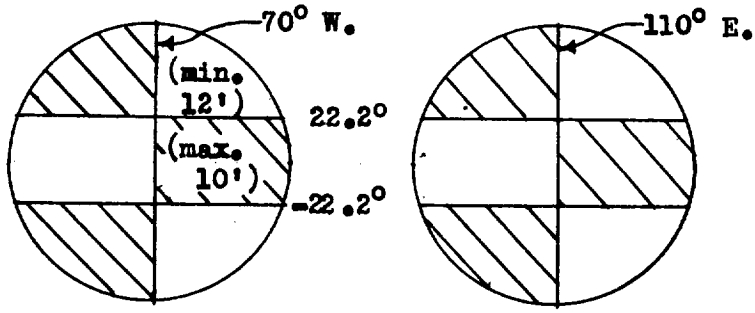
(figure 9)
J_{41} Deviations

G) Deviations giving rise to the J_{42} term of (80):

The J_{42} geopotential term is: (ref. 8)

$$-(-.062\times10^{-6})(\bar{a}/r)^4(1/8)\ 420\sin^2L - 60\ \cos^2L\ \cos 2(\phi - 65.2)\ .$$

Deviations are calculated as in B) above, the results appearing on figure 10 below.



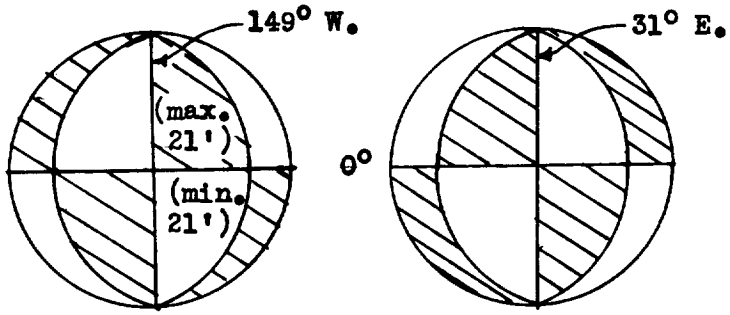
(figure 10)
J_{42} Deviations

H) Deviations giving rise to the J_{43} term of (80):

The J_{43} geopotential term is: (ref. 8)

$$-(-.035\times10^{-6})(\bar{a}/r)^4(1/8)\left[840\sin nL\right]\left[\cos^3 L\ \cos 3(\phi - 0.5^\bullet)\right] \quad .$$

Deviations are calculated as in B) above, the results appearing on figure 11 below.



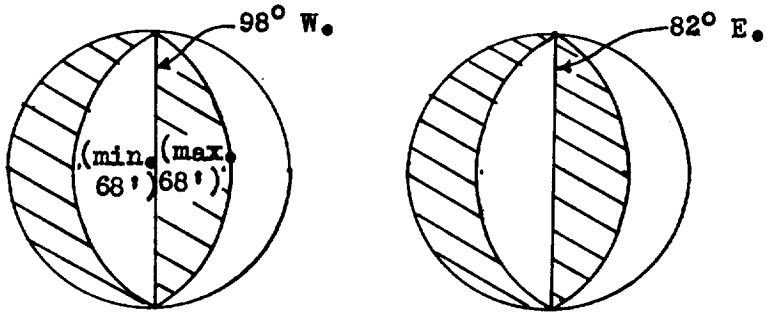
(figure 11)
J_{43} Deviations

I) Deviations giving rise to the J_{44} term of (80):

The J_{44} geopotential term is: (ref. 8)

$$-(-.031\times10^{-6})(\bar{a}/r)^4(1/8)\ 840\cos^4 L\ \cos 4(\phi - 14.9^\bullet) \quad .$$

Deviations are calculated as in B) above, the results appearing on figure 12 below.



(figure 12)
J_{44} Deviations

The complete geocentric-geographic gravity potential of (80), with the coefficients as reported in ref. 8, is given below. The longitude (ϕ) reference is greenwich with positive angles to the east. The radial(r) reference is the center of the model earth triaxial ellipsoid. The latitude (L) reference is the model ellipsoid's equatorial plane.

$$V = \frac{GM}{r}\Big[1 - (1082.48\times10^{-6})(\bar{a}/r)^2(1/2)(3\sin^2 L - 1) - \ldots\ldots$$
$$\ldots\ldots - (-1.2\times10^{-6})(\bar{a}/r)^2 3\cos^2 L \cos 2(\phi \neq 26.4^\circ) - (-2.56\times10^{-6}(\bar{a}/r)^3 \ldots$$
$$\ldots\ldots(1/2)(5\sin^3 L - 3\sin L) - (-1.9\times10^{-6})(\bar{a}/r)^3 \cos L(15\sin^2 L - 3)\ldots$$
$$\ldots\ldots\cos(\phi - 4.6^\circ) - (-.14\times10^{-6})(\bar{a}/r)^3 15\cos^2 L \sin L \cos 2(\phi \neq 16.8^\circ) - \ldots$$
$$\ldots\ldots - (-.10\times10^{-6})(\bar{a}/r)^3 15\cos^3 L \cos 3(\phi - 42.6^\circ) - (-1.84\times10^{-6}(\bar{a}/r)^4 \ldots$$
$$\ldots(35/8)(\sin^4 L - 30\sin^2 L \neq 3) - (-.52\times10^{-6})(\bar{a}/r)^4(1/8)(140\sin^3 L - \ldots$$
$$\ldots - 60\sin L)\cos L \cos(\phi - 237.5^\circ) - (-.062\times10^{-6}(\bar{a}/r)^4(1/8)(420\sin^2 L - \ldots$$
$$\ldots - 60)\cos^2 L \cos 2(\phi - 62.5^\circ) - (-.035\times10^{-6})(\bar{a}/r)^4(1/8)840\sin L \ldots$$
$$\ldots\ldots\cos^3 L \cos 3(\phi - 0.5^\circ) - (-.031\times10^{-6}(\bar{a}/r)^4(1/8)840\cos^4 L \cos 4(\phi - 14.9^\circ)\Big] .$$

.....(88)

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